Introduction

Here is a collection of counting problems. Questions and suggestions are welcome at `per.w.alexandersson@gmail.com`.

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Trimmed down for CSCI 200!

Exclusive vs. independent choice. Recall that we *add* the counts for exclusive situations, and *multiply* the counts for independent situations. For example, the possible outcomes of a dice throw are exclusive:

$$(\text{Sides of a dice}) = (\text{Even sides}) + (\text{Odd sides})$$

The different outcomes of selecting a playing card in a deck of cards can be seen as a combination of *independent* choices:

$$(\text{Different cards}) = (\text{Choice of color}) \cdot (\text{Choice of value}).$$

Labeled vs. unlabeled sets is a common cause for confusion. Consider the following two problems:

- Count the number of ways to choose 2 people among 4 people.
- Count the number of ways to partition 4 people into sets of size 2.

In the first example, it is understood that the set of chosen people is a *special* set — it is the *chosen set*. We choose two people, and the other two are not chosen. In the second example, there is no difference between the two couples. The answer to the first question is therefore

$$\binom{4}{2}, \qquad \text{counting the chosen subsets: } \{12, 13, 14, 23, 24, 34\}.$$

The answer to the second question is

$$\frac{1}{2!}\binom{4}{2}, \qquad \text{counting the partitions: } \{12|34, 13|24, 14|23\}.$$

That is, the issue is that there is no way to distinguish the two sets in the partition. However, now consider the following two problems:

- Count the number of ways to choose 2 people among 5 people.
- Count the number of ways to partition 5 people into a set of size 2 and a set of size 3.

In this case, the answer to both questions is $\binom{5}{3}$. The reason for this is that we can distinguish between the two sets in the partition in the second question, for example, *the set of size* 2 is unique.

We always consider people to be unique, and therefore labeled. This means that in a group of n people, we can talk about the first person, the second person, and so on. In a group of n *identical* objects, there is no á priori notion of the first object.

Overview of formulas

Every row in the table illustrates a type of counting problem, where the solution is given by the formula. Conversely, every problem is a *combinatorial interpretation* of the formula. In this context, a *group* of things means an unordered set.

Problem	Type	Formula
Choose a group of k objects from n different objects	Binomial coefficient	$\binom{n}{k}$
Partition n different objects into m labeled groups, with k_i elements in group i	Multinomial coefficients	$\binom{n}{k_1, \ldots, k_m}$
Partition n different objects into k non-empty groups, where there is no order on the sets	Partitions, Stirling numbers	$S(n,k)$
Partition n different objects into k labeled groups (which could be empty)	Multiplication principle	k^n
Partition n identical objects into m labeled groups	Dots and bars	$\binom{n+m-1}{m-1}$
Same, but with non-empty groups	Dots and bars	$\binom{n-1}{m-1}$
Order n different objects	Permutations	$n!$
Choose and order k different objects from n different objects	Permutations	$\frac{n!}{(n-k)!}$
Choose and order n objects, where there are k_i identical objects of type i	Multinomial coefficients	$\binom{n}{k_1, \ldots, k_m}$
Choices for (X,Y) if there are x choices for X and, independently, y choices of Y	Multiplication-principle	$x \cdot y$
Number of elements in $A \cup B \cup C$	Inclusion-exclusion	$\lvert A\cup B\cup C\rvert = \lvert A\rvert + \lvert B\rvert + \lvert C\rvert - \lvert A\cap B\rvert - \lvert A\cap C\rvert - \lvert B\cap C\rvert + \lvert A\cap B\cap C\rvert$

Binomial- and multinomial coefficients

Whenever $n \geq 0$ and $0 \leq k \leq n$, we define the *binomial coefficients* as

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}. \qquad \text{(Binomial coefficients)}$$

Sometimes the notation $C(n,k)$ for $\binom{n}{k}$ is used.

The binomial coefficients satisfy the following recursion:

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}. \qquad (1)$$

To choose k objects among $\{1,2,\dots,n\}$, we either exclude n, and choose k objects among $\{1,2,\dots,n-1\}$ *or* we include n, and choose additional $k-1$ objects among $\{1,2,\dots,n-1\}$.

We have the *binomial theorem*:

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} \qquad \text{(Binomial theorem)}$$

A generalization of the binomial coefficients are the *multinomial coefficients*. Whenever $k_1 + k_2 + \dots + k_r = n$, they are defined as

$$\binom{n}{k_1, k_2, \dots, k_r} = \frac{n!}{k_1! \cdots k_r!}. \qquad \text{(Multinomial coefficients)}$$

Stirling numbers

The Stirling numbers $S(n,k)$ can be computed recursively via a table, where every row is obtained from the previous via

$$S(n,k) = S(n-1,k-1) + k \cdot S(n-1,k).$$

and using the fact that $S(n,1) = S(n,n) = 1$.

Proof: To partition $\{1,2,\cdots,n\}$, into k groups, we either let n be in its own group, and partition $\{1,2,\dots,n-1\}$ into $k-1$ groups, or we partition $\{1,2,\dots,n-1\}$ into k groups and choose which of the k groups n belongs to.

Counting problems

Problem. 1
You are creating a 4-digit pin code. How many choices are there in the following cases?

(a) With no restriction.

(b) No digit is repeated.

(c) No digit is repeated, digit number 3 is a 0.

(d) No digit is repeated, and they must appear in increasing order.

(e) No digit is repeated, 2 and 5 must be present.

Problem. 2
How many shuffles are there of a deck of cards, such that $A\heartsuit$ is not directly on top of $K\heartsuit$, and $A\spadesuit$ is not directly on top of $K\spadesuit$?

A standard deck has 52 cards, divided into four suits ($\heartsuit$, $\spadesuit$, $\diamondsuit$, $\clubsuit$). There are 13 cards of each suit, $2,3,\dots,10$, J, Q, K, A, the Jack, Queen, King and Ace

Problem. 3
How many different words can be created by rearranging the letters in `SELFIESTICK`?

Problem. 4

How many flags can we make with 7 stripes, if we have 2 white, 2 red and 3 green stripes?

Problem. 5

We have four different dishes, two dishes of each type. In how many ways can these be distributed among 8 people?

Problem. 6

In how many ways can 8 people form couples of two?

Problem. 7

We go to a pizza party, and there are 5 types of pizza. We have starved for days, so we can eat 13 slices, but we want to sample each type at least once. In how many ways can we do this? Order does not matter.

Problem. 9

How many integer solutions does $x_1 + x_2 + \cdots + x_n = r$ have, with $x_i \geq 0$?

Problem. 10

How many integer solutions does the equation

$$x_1 + x_2 + x_3 + x_4 = 15$$

have, if we require that $x_1 \geq 2$, $x_2 \geq 3$, $x_3 \geq 10$ and $x_4 \geq -3$?

Problem. 11

How many integer solutions are there to the system of inequalities

$$x_1 + x_2 + x_3 + x_4 \leq 15, \qquad x_1, \ldots, x_4 \geq 0?$$

Problem. 13

Compute the number of surjections $f : A \to B$ if $|A| = n$ and $|B| = k$.

Problem. 16

How many words can you create of length 6, from the letters `a`, `b`, `c` and `d` if

- you must include each letter at least once, and
- `a` must appear exactly once.

Problem. 19

How many words can be made by rearranging `aabbccdd`, such that no `'a'` appears somewhere to the right of some `'c'`?

Reading comprehension

To see the intricacies in combinatorial reasoning, we now review a variety of counting problems.

Try to identify which of these choices allow for *repetition*, and which are ordered and unordered. Before proceeding, review the difference between *labeled* and *unlabeled* sets.

Words such as line, queue, list and shelf indicate an order, while words such as set, group, pile and bag indicate unordered arrangements. Additionally, people are always considered unique — no two persons are alike and they have names. You need to be aware if there are several sets, queues or groups involved: The two sets

$$\{\{a, b\}, \{c, d\}\} \text{ and } \{\{d, c\}, \{a, b\}\}$$

are considered equal. However, the two arrangements (with *named* sets)

$$A = \{a, b\}, B = \{c, d\} \text{ and } A = \{d, c\}, B = \{a, b\}$$

are considered different. Note that this intricacy can only occur for sets (or lists) of equal sizes.

Problem. 42

There are 8 people available. Count the number of ways

(a) to choose 6 of them and arrange them in a line.

(b) to choose 6 of them and place them into lines named A and B, with 3 in each.

(c) to choose 6 of them and place them into two equal-sized unlabeled lines.

(d) to choose 6 of them to make a group.

(e) to choose 6 of them and place them into groups named A and B, with 3 in each.

(f) to choose 6 of them and make two equal-sized unlabeled groups.

(g) to choose 6 of them and make three equal-sized unlabeled groups.

Problem. 43

There are 8 red balls available[1]. Count the number of ways

[1] These are identical!

(a) to choose 6 of them and arrange them in a line.

(b) to choose 6 of them to make a group.

(c) to choose 6 of them and give them to three people, some might not get any.

(d) to choose 6 of them and give them to three people, each person get at least one.

(e) to choose 6 of them and make three non-empty (unlabeled) groups.

(f) to choose 6 of them and divide them into piles.

Problem. 44

There are 8 types[2] of cookies available in a store. Count the number of ways

[2] This indicates that repetition is allowed — the same type can be used several times

(a) to pick 6 of them and arrange them in a line.

(b) to pick 6 of them and place them into lines named A and B, with 3 in each.

(c) to pick 6 of them and place them into two equal-sized unlabeled lines.

(d) to pick 6 of them to make a group.